Graceful Transitions through Maximal Persistence of Behavior ⋆

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Abstract: Problems of locomotion typically encounter switching between gaits. In many applications, it is desirable to make such transitions between modes or gaits inconspicuous and graceful. This is achieved by keeping the typical behavior of the system as persistent as possible. It is shown that this problem can be defined in a more abstract sense, and therefore generalized to a new class of problems, called "maximum persistence of behavior."
In this presentation, I will give a precise definition of this concept, for signals and for systems, and illustrate some applications from thermodynamics to signal study and robotics. I will also introduce the dual problem of filtering in the context of "maximal persistence of behavior." The results stem from joint work with Deryck Yeung (Trinity University, San Antonio, TX), Basit Memon (Habib University, Karachi, PK), and Vishal Murali (now at NVIDIA) while at Georgia Tech.
The paradigm starts by defining typical behaviors in the framework initiated by J.C. Willems. Gluing two typical signals or typical system behaviors requires connections (raccordations) that belong to a larger class of behaviors, but are locally close to the typical behaviors. Solutions with persistent behavior are then sought via optimization for a kernel or an image problem. This solution encapsulates the controllability problem of Willems, morphing theory in image analysis, object shaping, quasi-stationary transitions from thermodynamics, and orbit transfer problems, and can be characterized in a geometric way. In the robotics framework, obvious applications are in gait transitions for biomimetic robots as when switching from walking (e.g., while inspecting) to running (evasion), or a gait transition necessitated by a change in terrain (solid, muddy, or granular). In particular, I will discuss a method of smoothly connecting periodic orbits of systems. We first deal with the case when the periodic signals to be connected are of the same period and then move on to discuss the case of different periods, requiring in addition a frequency warping.

Keywords: Optimal Control, Behavioral Theory, Persistence

1. INTRODUCTION

In this presentation, a new method is proposed to switch between two signals belonging to a well-defined class (say constant, harmonic, periodic) while staying as close as possible to signals belonging to the given class. As such we can think of a generalization of a problem posed by H. Whitney. This then leads to switching between typical modes of operation of a system, (e.g. a steady states, where input and states are constant, or operation in a periodic regime). A first task is then to make precise what is meant by persistence of behavior. We understand gracefulness as making a smooth transition over a period of time, rather than just "throwing a switch" from one instance of the behavior to another. Surely, the concept is familiar in musical and dance transitions. The problem was first introduced in Verriest and Yeung [2008] and has been named the Gluskabi raccordation problem, for its suggestiveness to a mythical figure, Gluskabi, of the Penobscot Native tribe. A raccordation is also a term from civil engineering (from the French: to connect by a rope) and is used to denote a piece of a parabolic hyperboloid that connects two surfaces at different slopes. The key is that this surface is a ruled surface (through each point pass two straight lines), and is therefore easily constructed (Think also cooling towers.) Gradually, the local slope is warped between the two bordering slopes. The impetus to this study came from the paradoxical concept of *quasi-stationary transitions* in thermodynamics (See Andresen et al. [1977], Berry et al. [2000], De Vos and Soete [2000]). Subsequently, the problem was studied more extensively in the Ph.D. work of Deryck Yeung Yeung [2011], Abdul-Basit Memon Memon [2014] and Vishal Murali Murali [2021]. Prior work on connecting periodic orbits has also been studied by Sultan [2007, 2008], Sultan and Kalmar-Nagy [2011]. The methods we propose are based on the kernel and image methods from behavioral system theory (See Polderman and Willems [1998]).

We briefly outline some of the practical applications of this problem. Most animals have natural periodic gaits for locomotion. (See Golubitsky and Stewart [2003],Haynes et al [2006]). However, it is difficult to switch between these gaits instantaneously primarily due to inertia. The

⋆ This work was partially supported by NSF grant CPS 1544857.

raccordation problem seeks to make a transition between these two distinct periodic gaits in a graceful manner over a given interval of time. (See Murali et al. [2019]). This has already been applied to obtain graceful gait transitions for a biomimetic worm in Memon and Verriest [2014b]. We also hypothesized that the same idea could be used to obtain graceful transitions of different walking gaits of bipedal robots, for instance between a walking gait on a hard ground such as concrete and a gait on a soft ground with granular media such as sand. This was explored in Murali et al. [2018, 2020]. Other applications include image morphing, filtering and the design of a perfect chirp for linear systems.

This paper gives the underpinnings of our theoretical approach. First, we review the requisite terminology from behavioral approach to system theory to set the stage for our approach. We then formulate the Gluskabi problem, first for signal spaces, then for system behaviors.

2. BACKGROUND MATERIAL

Our approach follows the behavioral theory developed by Jan Willems. The reader can refer to Willems [2007] for more information on this approach to system theory. A *dynamical system* is a triple $\sum = (\mathbb{T}, \mathbb{W}, \mathcal{B})$. Here $\mathbb{T}$ is the time set, it could be for instance continuous $\mathbb{T} = \mathbb{R}$ or discrete $\mathbb{T} = \mathbb{Z}$. $\mathbb{W}$ is the signal space and we denote by $\mathbb{W}^{\mathbb{T}}$ the set of all functions $w : \mathbb{T} \to \mathbb{W}$. The set $\mathcal{B}$ is a subset of $\mathbb{W}^{\mathbb{T}}$ and is called the *behavior*. It can be thought of as the set of allowable trajectories of the system.
If the behavior is *linear*, then $\mathbb{W}$ is a vector space over $\mathbb{R}$ and if $w_1 \in \mathcal{B}$ and $w_2 \in \mathcal{B}$ then $\alpha_1 w_1 + \alpha_2 w_2 \in \mathcal{B}$ for $\alpha_1, \alpha_2 \in \mathbb{R}$. We thus assume $\mathbb{T} = \mathbb{R}$ and $\mathbb{W} = \mathbb{R}^n$. The system $\sum$ is *time invariant* if for every $w \in \mathcal{B}$ and every $\tau \in \mathbb{R}$, ($\mathbf{T}_\tau$ is the shift operator) the function $(\mathbf{T}_\tau)w = w(t + \tau)$ is an element of $\mathcal{B}$. The LTI behavior $\sum$ is *controllable* if for any two trajectories w_1 and w_2 in $\mathcal{B}$, there exists a trajectory $w \in \mathcal{B}$ such that

$$\sigma_t(w) = \begin{cases} \sigma_t(w_1) & \text{if } t \le 0 \\ \sigma_t(w_2) & \text{if } t \ge \tau \end{cases}$$

where σ_t denotes the evaluation function given by $\sigma_t(w) = w(t)$ for all $w \in \mathbb{W}^{\mathbb{T}}$. This is a slight departure from Willems's definition, which better suits our work, but if $w_2 \in \mathcal{B}$, then so is its translation since the behavior is time-invariant.

2.1 Persistence of behavior

See Memon [2014] for further explanations and examples on some of the definitions to follow. We first define the *universum*, the behavior $(\mathbb{T}, \mathbb{W}, \mathcal{U})$ which is the set of all admissible signals. For instance, $\mathcal{B}_0$ could be $C(\mathbb{R}, \mathbb{W})$, the set of continuous signals or C^∞, the smooth signals or piecewise continuous, L^2_{loc}, etc. The choice depends on the applications. With a particular system in mind, we can also define the *base system* as the behavior with $w^\top = [x^\top, u^\top]$ with the dynamic constraint $\dot{x} = Ax + Bu$. In operator form, such behavior is presented as the kernel of the operator $\mathbf{Op} = [\mathbf{D} - A, -B]$. The base system corresponds to a hard constraint the relevant signals (here input and state) need to adhere to. For signal behaviors, the base behavior is the universum, while in the dynamic case, it is further restricted.

We consider, $\mathbb{T} = \mathbb{R}$, $\mathbb{W} = \mathbb{R}^n$ and $\mathcal{B} = C^{2n-2}(\mathbb{R}, \mathbb{R}^n) \cap \ker(\mathbf{Op})$ where $C^{2n-2}(\mathbb{R}, \mathbb{R}^n)$ denotes set of $2n - 2$ times continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}^n$ and $\mathbf{Op}$ defines the law of the system (in a broad sense). Note that $\mathcal{B}$ is a strict subset of $\mathbb{W}^T$ which contains all functions (not necessarily continuous) $w : \mathbb{R} \to \mathbb{R}^n$. We also remark that the solution of the raccordation problem depends on the smoothness conditions. The raccordation problem seeks to connect two trajectories of a desired type. To do this, we need the following definition:

Definition 2.1. A type, $\mathcal{T}$, is a sub-behavior of the universum (in the signal case) or the base behavior in the dynamic case, such that all its elements share a (typical) quality, specified either as the kernel of an operator $\widehat{\mathbf{Op}} : \mathcal{B} \to \mathcal{V}$, where $\mathcal{V}$ is a set of functions $v : \mathcal{R} \to \mathcal{R}^m$ such that $\mathcal{V}|_{\mathcal{I}}$ can be made into a normed space for bounded intervals $\mathcal{I}$.
Thus, in the kernel representation, a type is specified by

$$\mathcal{T} = \{\, w \in \mathbb{B}_0 \,|\, \widehat{\mathbf{Op}} w = 0 \,\}.$$

Of course one should take care in choosing $\widehat{\mathbf{Op}}$, as for any linear operator $\widehat{\mathbf{Op}}_1$ it holds that the kernel of $\widehat{\mathbf{Op}}_1 \widehat{\mathbf{Op}}$ contains the kernel of $\widehat{\mathbf{Op}}$, but typically contains more signals that $\ker \widehat{\mathbf{Op}}$. Hence $\widehat{\mathbf{Op}}$ should be "minimal" in describing the signals of interest.
Alternatively, the type can be specified as the image of an injective mapping from a parameter set Θ to the universum. If ρ is a parameterization of a type: $\rho : \Theta \to \mathcal{B}_0$, then

$$\mathcal{T} = \operatorname{Im} \rho = \{ w \in \mathcal{B}_0 \,|\, \exists\, \theta \in \Theta \; s.t. \; \rho(\theta) = w \,\}.$$

We shall see further that it is of interest to endow Θ with a metric.
For instance, the type of constant scalar signals, $\mathcal{T}_c$ has kernel representation $\mathcal{T}_c = \{w | \mathbf{D}w = 0\}$ and image representation $\mathcal{T}_c = \operatorname{Im} \rho_c$, where $\rho_c : \mathbb{R} \to \mathcal{B} : r \to w$ with $\sigma_t(w) = r \,\forall t$.

Definition 2.2. A trait, $\mathcal{T}_\theta$, is a particular behavior (thus an element) of a given type. It is the image of a singleton, $\{\theta\} \subset \Theta$ in the image representation, or a unique element (e.g., defined by a pinning condition, $\sigma(w) = w_0 \in \mathbb{R}^n$) in the kernel representation.

We assume that $\mathcal{V} = \mathcal{L}^2_{loc}(\mathbb{R}, \mathbb{R}^k)$. Hence, $\mathcal{V}|_{\mathcal{I}}$ is a Hilbert space for compact connected intervals $\mathcal{I}$ when endowed with the inner product $\langle f, g \rangle = \int_{\mathcal{I}} f^\top(t) g(t) \, \mathrm{d}t$. Let us denote by $\gamma : \mathcal{V} \to \mathcal{V}|_{\mathcal{I}}$ the (restriction) operator that takes $f \in \mathcal{V}$ and returns $\gamma(f) = f|_{\mathcal{I}}$.

Definition 2.3. Given an arbitrary subset $\mathcal{A} \subset \mathcal{B}_0$ of behaviors, and a type $\mathcal{T}$, specified as the kernel of an operator $\widehat{\mathbf{Op}}$. An element $\overline{w} \in \mathcal{A}$ is said to be *maximally persistent of type* $\mathcal{T}$ in the interval $\mathcal{I} = [a, b]$ if

$$\overline{w} = \operatorname{argmin}_{w \in \mathcal{A}} \| \widehat{\mathbf{Op}}\, w_{\mathcal{I}} \|.$$

By definition, if $\mathcal{A} \subseteq \mathcal{T}$, then all elements, w of $\mathcal{A}$ are maximally persistent of type $\mathcal{T}$, and $\| \widehat{\mathbf{Op}}\, w_{\mathcal{I}} \| = 0$.

Periodicity defined by the shift-operator has non-smooth elements in its kernel. However recall that the smoothness is induced since we assumed that $\mathcal{T} \subset \mathcal{B}_0 = C(\mathbb{R}, \mathbb{R})$.

3. SOME INTERESTING TYPES

Before presenting results about Gluskabi raccordations, three interesting types are introduced for scalar behaviors. The first is the stationary (or constant) type, $\mathcal{T}_c$, defined by the operator $\mathbf{D}$. The second is the harmonic type of frequency ω, $\mathcal{T}_{h(\omega)}$, answering to $\mathbf{D}^2 - \omega^2$. The third is the smooth periodic type of period T defined by the shift operator $\mathbf{T}_{\frac{T}{2}} - \mathbf{T}_{-\frac{T}{2}}$ (See Verriest [2021]). The symmetry is not required to define $\mathcal{T}_{p(T)}$, but will help later. The corresponding parameter spaces for the image representation are $\Theta_c = \mathbb{R}$, $\Theta_h = \mathbb{C}$, and $\Theta_p = \ell_2(\mathbb{C})$, of course, with $\omega = \frac{2\pi}{T}$ understood in the latter cases. These parameterizations are the familiar phasor and complex Fourier representation.

These ideas can be generalized in several directions: First, the extensions of constant, harmonic and periodic type to vector behaviors are accomplished by the component-wise action of the operator. Another type generalizes $\mathcal{T}_c$ and $\mathcal{T}_h$, and is defined by a more general linear time-invariant differential matrix polynomial operator, $a(\mathbf{D})$, with $a \in \mathbb{R}^{n\times k}[\mathbf{D}]$. Denote this type by $\mathcal{T}_{a(\mathbf{D})}$. Different traits are then specified by k pinning conditions for each of the n components. The following theorem shows that a set of types generates bigger types.

Theorem 1
Let $\mathcal{T}_i$, i=1,2, be two types associated respectively by the operators $\widehat{\mathbf{Op}}_1$ and $\widehat{\mathbf{Op}}_2$. Then the Minkowsky sum $\mathcal{T}_1 \oplus \mathcal{T}_2$ is a type associated with the operator llcm$\,(\widehat{\mathbf{Op}}_1, \widehat{\mathbf{Op}}_2)$, where llcm$\,(\mathbf{A}, \mathbf{B})$ denotes the *left least common multiple* of the operators $\mathbf{A}$ and $\mathbf{B}$. If the operators commute, this is simply their product.
Proof: Follows from Bezout's theorem. □

3.1 Fixed LTI Types

The behavior type generated by $\mathbf{D} + a$ and $\mathbf{D} + b$ is the kernel of the second order operator $(\mathbf{D} + a)(\mathbf{D} + b)$. The behavior generated by $\mathcal{T}_c$ and the time-variant behavior $\mathcal{T}_{\mathbf{D}^2+t^2+1}$ is $\mathcal{T}_{\mathbf{D}^2+(t^2+1-\frac{2t}{t^2+1})}$.
It follows that the type of linear combinations of a set of N exponentials will correspond with the kernel of an N-th order LTI differential operator. Incidentally, each Fourier component of a periodic function satisfies a harmonic autonomous ODE, so that the approximating finite sum of the components (up to frequency $N\omega$), must be in the kernel of an infinite product operator (See Sato et al. [1983]):

$$w \in \mathcal{T}^{(N)}_{p(\omega)} \quad \Leftrightarrow \quad \mathbf{D}\prod_{k=1}^{N}(\mathbf{D}^2 + k^2\omega^2)w = 0. \tag{1}$$

This does not quite converge to a well defined operator (see Silverman [1984]) as $N \to \infty$, but the limiting type, $\mathcal{T}_{p(\omega)}$ is characterizable as the kernel of the infinite product operator (see Fan and Hao [1994]):

$$\widehat{\mathbf{Op}}_{\mathrm{p}} = \mathbf{D}\prod_{k=1}^{\infty}(1 + \frac{1}{k^2\omega^2}\mathbf{D}^2). \tag{2}$$

This can also be expressed as $\widehat{\mathbf{Op}}_{\mathrm{p}} = \sinh\left(\frac{\pi}{\omega}\mathbf{D}\right)$.
Obviously, trying to extend this to the set of exponentials with as exponent any α in an interval will fail, as this set is not denumerable. But how can one define "first order behavior" then? Said differently, what characterizes the set of all real exponentials?

3.2 Linear Time-Invariant Differential (LTID) Type

Let $\mathcal{L}^k_n$, of some order n, be the set of all solutions to any system of k constant coefficient homogeneous differential equations of nth order. This type was first introduced in Verriest [2012a], where the operator was derived for the scalar n-th order differential equation case, i.e., when $k = 1$, and the operator for the vector case was later derived in Memon and Verriest [2014a]. This behavior is represented compactly by a square polynomial matrix, $R \in \mathbb{R}[\xi]^{k\times k}$, where the highest degree of the polynomials is n,

$$R(\xi) := R_0\xi^n + R_1\xi^{n-1} + \cdots + R_n\xi.$$

The idea is illustrated first for the simplest type in this category. Consider the type comprised of solutions, w, to the first-order ODE, $\dot{w}(t) - a\,w(t) = 0$, associated with the operator $\widehat{\mathbf{Op}} = \mathbf{D} - a\mathbf{I}$. This is the type of all scalar multiples of the exponential e^{at}. The LTID type $\mathcal{L}^1_1$ is the type containing solutions to *any* monic first-order ODE, i.e., the solutions to the previous differential equation for any value of a. Note that if $w \in \mathcal{L}^1_1$, then

$$\frac{\dot{w}}{w} = a \qquad \text{for some } a \in \mathbb{R}. \tag{3}$$

Consequently, constancy of a is a characterization of first-orderness. Differentiating this equation gives,

$$\frac{\ddot{w}w - \dot{w}^2}{w^2} = 0. \tag{4}$$

i.e., the type $\mathcal{L}^1_1$, is given by the kernel of the *nonlinear* operator, $\widehat{\mathbf{Op}}$, such that $\widehat{\mathbf{Op}}\,w = \ddot{w}w - \dot{w}^2 = \begin{vmatrix} w & \dot{w} \\ \dot{w} & \ddot{w} \end{vmatrix}$, which is the Wronskian of the functions w and $\dot{w}$, characterizing the type $\mathcal{L}^1_1$. This is generalized in the following theorem.

Theorem 2
The LTID type, $\mathcal{L}^1_n$, is characterized by the kernel of a Wronskian operator, $\mathbf{W}$, specifically the operator associated with the type is defined as

$$\widehat{\mathbf{Op}}\,w = \mathbf{W}\left(w, \cdots, w^{(n)}\right) = \begin{vmatrix} w & \dot{w} & \cdots & w^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ w^{(n)} & w^{(n+1)} & \cdots & w^{(2n)} \end{vmatrix}.$$

Proof: Say it's given that $\widehat{\mathbf{Op}}\,w = 0$. It is known that if the Wronskian of a finite family of analytic functions is zero, then the functions are linearly dependent (See Bostan and Dumas [2010]). Therefore, assuming that w is analytic,

$$r_0w^{(n)} + r_1w^{(n-1)} + \cdots + r_{n-1}\dot{w} + r_nw = 0,$$

for some real coefficients $r_0, r_1, \cdots, r_n$. This means that w is an analytic solution to an nth order constant coefficient homogeneous differential equation. It is also known that any higher order differential equation of this type can be converted into a first order differential equation in vector form, the smooth solution to which is the matrix exponential times a constant vector. The matrix exponential is an analytic function and so all smooth solutions are actually analytic. This means that if $\widehat{\mathbf{Op}}\, w = 0$ then w is a smooth solution to some nth order differential equation and is subsequently contained in the LTID type $\mathcal{L}_n^1$.

In the other direction, if $w \in \mathcal{L}_n^1$, then there exist $r_i \in \mathbb{R}$ for $i \in \{0, 1, \cdots, n\}$ such that

$$r_0 w^{(n)} + r_1 w^{(n-1)} + \cdots + r_{n-1}\dot{w} + r_n w = 0. \quad (5)$$

This implies that the functions $\{w, \dot{w}, \cdots, w^{(n)}\}$ are linearly dependent and hence their Wronskian is zero, i.e. $\mathbf{W}\left(w, \dot{w}, \cdots, w^{(n)}\right) = 0.$ $\square$

The multivariable case is somewhat more technical.

Theorem 3

The LTID type, $\mathcal{L}_n^k$, is characterized by the kernel of the Schur complement of the Wonskian matrix $\widehat{\mathbf{W}}$ in $\mathbf{W}$, more specifically, the operator defining the type $\mathcal{L}_n^k$ is

$$\widehat{\mathbf{Op}}\, w = \mathrm{Schur}\left(\widehat{\mathbf{W}}\right) \text{ in } \mathbf{W},$$

where

$$\mathbf{W} = \left[\begin{array}{ccc|c} w & \cdots & w^{(nk-1)} & w^{(nk)} \\ \vdots & & \vdots & \vdots \\ w^{(n-1)} & \cdots & w^{(n-1+nk-1)} & w^{(n-1+nk)} \\ \hline w^{n} & \cdots & w^{(n-1+nk)} & w^{(n+nk)} \end{array}\right] \quad (6)$$

and

$$\widehat{W} = \begin{bmatrix} w & \cdots & w^{(nk-1)} \\ \vdots & & \vdots \\ w^{(n-1)} & \cdots & w^{(n-1+nk-1)} \end{bmatrix}. \quad (7)$$

Proof: See Memon [2014].

3.3 Mixed Kernel and Image Approach

As an alternative to the LTID approach, suppose that for each $\omega \in \mathbb{R}$, $\widehat{\mathbf{Op}} = a(\mathbf{D}; \omega)$ is an LTI operator specifying a trait $\mathcal{T}_\omega$. We are interested in the encapsulating behavior, $\mathcal{T}$, comprising all $\mathcal{T}_\omega$. A particular viewpoint to connect w_{initial} to w_{final} is to ramp ω from ω_{initial} to ω_{final} and assign the raccordation at time $t \in \mathcal{I}$ as the evaluation at t of the trajectory $w_{\omega(t)}$. It is seen that this corresponds to an extended operator representation acting on the augmented $w^{\text{ext}} = \begin{bmatrix} w \\ \omega \end{bmatrix}$:

$$\widehat{\mathbf{Op}}^{\text{ext}} w^{\text{ext}} = \begin{bmatrix} a(\mathbf{D}; \omega)\, w \\ \mathbf{D}\omega \end{bmatrix}.$$

But this is a nonlinear operator in ω. This simple approach is frequently used for raccordations of periodic signals with differing period (See Murali and Verriest [2017]). However, this is not entirely satisfying. If the behaviors are periodic, the low frequencies are tracked well, but not the high frequencies, as shown next.

3.4 Area law

If the parameter ω represents the frequency of a harmonic signal, it seems reasonable (in an absolute point of view)

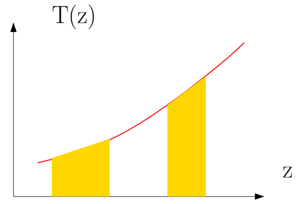

Fig. 1. Area law (T(z) is frequency).

that, if Δt is the time it takes to change ω by $\Delta\omega$, then the oscillation $\Delta\theta$ should be fixed (i.e., the same number of 'turns' in equal step-ups). This makes the change in ω per turn the least inconspicuous by keeping it fixed (See Murali and Verriest [2018]). But this means $\Delta\omega = \dot{\omega}\Delta t$ and $\Delta\theta = \omega\Delta t$. Hence in the limit

$$\frac{\Delta\omega}{\Delta\theta} \to \frac{\dot{\omega}}{\omega}$$

is constant, say c. Integrating $\dot{\omega} = c\omega$ with the boundary conditions in $[0, T]$ yields the law

$$\ln \omega(t) = \mathrm{Conv}\left[\ln \omega_{\text{initial}}, \ln \omega_{\text{final}}\right],$$

where $\mathrm{Conv}[\xi, \eta](t) = \xi\frac{(T-t)}{T} + \eta\frac{t}{T}$. If one graphs $\omega(t)$, then the law expresses that for fixed $\Delta\omega$, the area underneath $\omega(t)$ between $t_-(\omega)$ and $t_+(\omega)$ where $\omega(t_\pm) = \omega(t \pm \frac{1}{2}\Delta\omega)$ is covered, remains constant. See Figure (1) If for some reason the relative change of ω seems more appropriate to the problem, we set $\frac{\Delta\omega}{\omega} = \frac{\dot{\omega}\Delta t}{\omega}$ and $\Delta\theta = \omega\Delta t$, which leads to the ODE $\dot{\omega} = c\omega^2$ with c constant. This yields the law

$$\frac{1}{\omega(t)} = \mathrm{Conv}\left[\frac{1}{\omega_{\text{initial}}}, \frac{1}{\omega_{\text{final}}}\right].$$

3.5 Constrained LTID types

LTID types may be further constrained, for instance take the type of all harmonic functions (thus of any frequency), $\mathcal{T}_{\text{har}} = \bigoplus_{\omega \in \mathbb{R}_+} \mathcal{T}_{h(\omega)}$. The trait of harmonics is the behavior of all functions w such that it satisfies $\ddot{w} + \omega^2 w = 0$, for some $\omega \in \mathbb{R}$. Directly employing the ideas expounded in section 3.1, the type of harmonics could be defined by (8),

$$\frac{w\, w^{(3)} - \dot{w}\,\ddot{w}}{w^2} = 0. \quad (8)$$

but then the solution set also includes all hyberbolic sines and cosines (for which $\omega^2 < 0$). In order to completely characterize the type of harmonics, an additional condition is required, namely the coefficient of w must be positive or $\frac{\ddot{w}}{w} < 0$. Therefore, the trait of harmonics is characterized by,

$$\begin{cases} w\, w^{(3)} - \dot{w}\,\ddot{w} = 0 \\ \dfrac{\ddot{w}}{w} < 0 \end{cases}. \quad (9)$$

4. STATIC GLUSKABI RACCORDATIONS

We first state the *static (or signal-) raccordation problem*: Given a type $\mathcal{T}$, specified as the kernel of $\widehat{\mathbf{Op}}$, two traits $w_1 \in \mathcal{T}$ and $w_2 \in \mathcal{T}$, and a raccordation interval $\mathcal{I} = [a, b]$, find the behavior $w \in \mathcal{B}_0$ that is maximally persistent in $\mathcal{I}$, among all behaviors with the following restrictions:

$$\sigma_t(w) = \begin{cases} \sigma_t(w_1) & \text{if } t \le a \\ \sigma_t(w_2) & \text{if } t \ge b. \end{cases} \quad (10)$$

The class of behaviors satisfying (10), specified by the traits w_1 and w_2 and the interval $\mathcal{I}$, will be denoted by $\mathcal{A}_{\mathcal{I}}(w_1, w_2)$. Obviously, for (10) to make sense, $\mathcal{V}$ must be a normed space. Many norms are possible, and a judicious choice is needed to pick the one that is relevant for the problem at hand. For example, suppose the signals to be connected are $w_1(t) \in \mathcal{B}_0$ and $w_2(t) = w_1(t)$ and the transition interval is $\mathcal{I} = [a, b]$. It is clear that $w = w_1(t)$ is the solution as $g(\hat{\mathbf{Op}}w) = 0$ and w trivially satisfies the boundary conditions. If the behavior is a linear time-invariant behavior, a sufficient condition for the existence of a raccordation for arbitrary $w_1, w_2 \in \mathcal{T}$ is that the extended behavior be controllable (in the sense of behavioral theory). The rationale behind this is that to connect two traits in $\mathcal{T}$ one has to leave the type $\mathcal{T}$ in the raccordation interval as $\mathcal{T}$ is not controllable. Hence enlarge the class of behaviors to the class of controlled behaviors given by $\widehat{\mathbf{Op}}\, w = u$, where $u \in \mathcal{U}$. The *least conspicuous* control is then the one that keeps u as close as possible to zero. The problem of maximal persistence is then an optimal control problem

$$\min_{w \in \mathcal{A}_{\mathcal{I}}(w_1, w_2)} \|\widehat{\mathbf{Op}}\, w\|^2. \tag{11}$$

subject to gluing conditions at the boundaries (see further).

Definition 2.4: Gluskabi map Given a type $\mathcal{T}$ and a raccordation interval $\mathcal{I} = [a, b] \subset \mathbb{T}$, the basic Gluskabi map $g : \mathcal{T} \times \mathcal{T} \to \mathcal{B}_0$ with respect to the norm $\|\cdot\|$ is defined by

$$g(w_1, w_2)(t) = \overline{w}, \tag{12}$$

$$g(w_1, w_2)(t) = \begin{cases} \sigma_t(w_1) & \text{if } t \le a \\ \sigma_t(\overline{w}) & \text{if } a < t < b \\ \sigma_t(w_2) & \text{if } t \ge b. \end{cases} \tag{13}$$

where $\overline{w}$ solves the raccordation problem in (11).

4.1 Gluskabi raccordations - Kernel Approach

Given a type $\mathcal{T}$, the Gluskabi extension of $\mathcal{T}$, denoted $\mathcal{G}_{\mathcal{T}}$ is the set of all Gluskabi extensions of $\mathcal{T}$.

Theorem 4
Given an interval $\mathcal{I} = [a, b]$ and type $\mathcal{T}$ specified by a linear operator $\widehat{\mathbf{Op}}$, its basic Gluskabi extension is the type associated with a *linear* operator $\widehat{\mathbf{Op}}^*\widehat{\mathbf{Op}}$, where $\widehat{\mathbf{Op}}^*$ is the adjoint of $\widehat{\mathbf{Op}}$.
Proof: This is a special case of Theorem 5 below.

The raccordation in the interval $[a, b]$ will be called the "*Gluskabi raccordation*." In view of Theorem 4, the set of Gluskabi extensions of type $\mathcal{T}$ is a new type denoted $\mathcal{G}_{\mathcal{T}}$. We note that $\mathcal{T} \subset \mathcal{G}_{\mathcal{T}}$. Since the space $\mathcal{V}$ restricted to $\mathcal{I}$ admits multiple norms, the Gluskabi map and extension depend on the chosen norm. For instance, if $\widehat{\mathbf{Op}}$ is a differential operator of order m, then a Sobolev norm of degree m can be used to get the required level of persistence. For instance, if $\widehat{\mathbf{Op}} : C^r(\mathbb{R}, \mathbb{R}) \to C^s(\mathbb{R}, \mathbb{R})$,

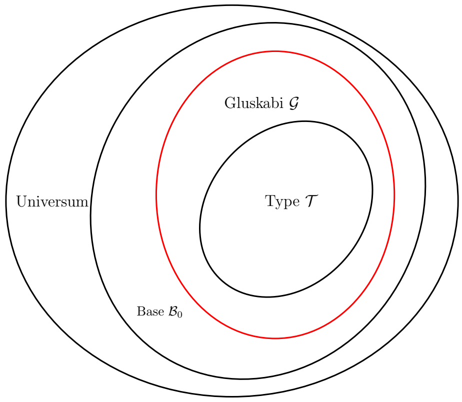

Fig. 2. Relation of Gluskabi Extension to other behaviors

then the Sobolev norm $\|\cdot\|_W$ of $u \in \mathcal{V} = C^s(\mathbb{R}, \mathbb{R})$ is defined by

$$\|u\|_W = \sum_{i=0}^{k} \rho_i \|\mathbf{D}^i u\|_2, \tag{14}$$

where $\rho_i > 0$, $k \le s$ and $\|v\|_2 = \int_a^b v^2(t)\, dt$.
This generalizes Theorem 4:

Theorem 5
Given an interval $\mathcal{I} = [a, b]$ and type $\mathcal{T}$ specified by a linear differential operator, $\widehat{\mathbf{Op}}$, of degree n. its Gluskabi extension for a Sobolev norm $\|\cdot\|_W$ in (14) is the type associated with a *linear* operator $\widehat{\mathbf{Op}}^* \rho(\mathbf{D}^2)\, \widehat{\mathbf{Op}}$, where $\widehat{\mathbf{Op}}^*$ is the adjoint of $\widehat{\mathbf{Op}}$ and

$$\rho(\mathbf{D}^2) = \sum_{i=0}^{k} (-1)^i \rho_i \mathbf{D}^{2i}.$$

Proof: Adjoining the Gluskabi constraint with Lagrange multiplier λ, the Hamiltonian for the LQ optimization problem is

$$H = \sum_{i=0}^{k} \frac{\rho_i}{2} \langle \mathbf{D}^i u, \mathbf{D}^i u\rangle + \langle \lambda, \widehat{\mathbf{Op}}\, w - u\rangle.$$

Using properties of the adjoint, the first variation is

$$\delta H = \sum_{i=0}^{k} \rho_i \langle (-\mathbf{D})^i \mathbf{D}^i u, \delta u\rangle + \langle \widehat{\mathbf{Op}}^* \lambda,\, \delta w\rangle - \langle \lambda, \delta u\rangle.$$

Choosing $\lambda = \sum_{i=0}^{k} \rho_i \langle (-1)^i \mathbf{D}^{2i} u = \rho(\mathbf{D}^2) u$ leaves

$$\delta H = \langle \widehat{\mathbf{Op}}^* \lambda, \delta w\rangle.$$

By the Dubois-Reymond lemma, a necessary condition for optimality is $\widehat{\mathbf{Op}}^* \lambda = 0$. Substituting the choice for λ yields

$$\widehat{\mathbf{Op}}^* \rho(\mathbf{D}^2)\, u = 0$$

and thus elements of $\mathcal{G}_{\mathcal{T}}$ are characterized by

$$\widehat{\mathbf{Op}}^* \rho(\mathbf{D}^2)\, \widehat{\mathbf{Op}}\, w = 0.$$

This is an ODE of order $2n + 2k$, and therefor a trait w has $2(n+k)$ degrees of freedom. This allows smooth gluing since n boundary conditions (on the successive derivatives of w) can be matched at both bounderies with the initial and final trait of $\mathcal{T}$. The remaining $2k$ degrees of freedom allow to make the 'control' u and its first $k-1$ derivatives equal to zero at the boundary, thus also gluing the control

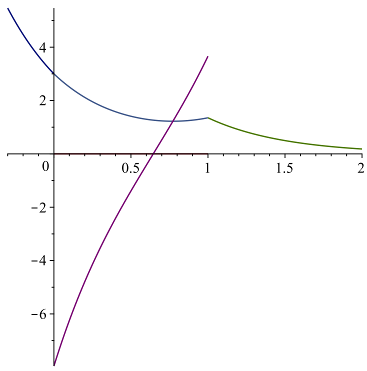

Fig. 3. Raccordation of exponential type (L_2 norm).

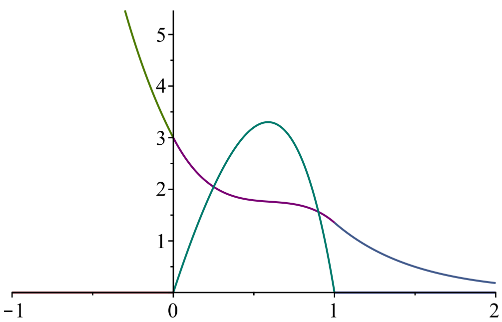

Fig. 4. Raccordation of exponential type (W_1 norm).

smoothly at the boundary. □

Example 1: Consider the Gluskabi raccordation of the exponentials $3\,\mathrm{e}^{-2t}$ and $10\,\mathrm{e}^{-2t}$ over the interval $[0, 1]$. The type is $\mathcal{T}_a = \{\, w \,|\, (\mathbf{D}+a)w = 0 \,\}$. Here $\widehat{\mathbf{Op}} = \mathbf{D}+a$ and $n = 1$. For the L_2 norm for $u = (\mathbf{D}+a)w_g$, the Gluskabi raccordation satisfies

$$(\mathbf{D}^2 - a^2)w_g = 0, \quad w_g(0) = 3, \quad w_g(1) = \frac{10}{e^2}.$$

Its solution is shown in Figure 3 together with the requisite control u (measuring a distance from $\mathcal{T}_a$). The gluing is not smooth and the control u looks uninteresting. However with a Sobolev norm $W_1[u]^2 = \|u\|^2 + \|\dot{u}\|^2$, the solution is smoother and the control u looks more appeasing (Figure 4). The solution is the convex combination:

$$g(w_{\text{initial}}, w_{\text{final}}) = w_- \frac{\sinh a(1-t)}{\sinh a} + w_+ \frac{\sinh at}{\sinh a}$$

where $w_- = w_{\text{initial}}(0)$ and $w_+ = w_{\text{final}}(1)$. Note that there is no freedom left to smoothly glue the control u to the zero value. This would have been possible had we chosen the Sobolev norm of degree 2.

In fact, it is easily shown that all Gluskabi raccordations specified by Theorem 5 are convex combinations for some function ϕ, monotone in the raccordation interval. Figure **??** shows the raccordation over $[0, 1]$ gluing the exponentials $3\exp(-2t)$ to $10\exp(-2t)$. We also display the control $u = \mathbf{D}(\frac{\dot{w}}{w})$.

4.2 Gluskabi raccordations - Image Approach

Likewise, in the image representation, one should pull the function norm back to the parameter space. Locally (near $t_0 \in \mathcal{I}$), a piece of the raccordation may approximate an element of type $\mathcal{T}$, say with parameter $\theta(t_0)$. This requires the definition of a local *parameter identification map*, $\gamma : \mathcal{T}|_{[t-\epsilon, t+\epsilon]} \to \Theta$, such that $\rho(\gamma(w)) = w$ for all $w \in \mathcal{T}$. The raccordation is then pulled back to a path in parameter space Θ from θ_1 to θ_2 for which the path length can be computed given the metric in Θ.

Consider a type, $\mathcal{T}$ (e.g. all LTI $n-th$ order behaviors). A particular trait may be specified by the roots of a particular monic differential polynomial $\widehat{\mathbf{Op}}$. The coefficients in $\widehat{\mathbf{Op}}$ comprise a paramaterization of the trait. As shown in Theorem 5, determining the parameter in Θ, here $\mathbb{R}^n$ for a signal w, amounts to a nonlinear operator, say $\mathbf{N}$ acting on the signal. We'll denote this parameter map (locally defined for each t) as $\mathbf{N}[w](t) = \theta$, where the square brackets remind us of absence of the nonlinear property. Now we are interested in the Gluskabi raccordation of two elements in $\mathcal{T}$ but belonging to different traits, specified by θ_i, $i = 1, 2$, in Θ. Let $\mathbf{N}$ be of nth order. Since all elements in $\mathcal{T}$ are specified by a fixed θ, it holds that

$$\mathbf{DN}[w] = 0.$$

Thus this qualifies as a (nonlinear) kernel representation of $\mathcal{T}$. We can now use similar ideas as in Theorem 5.

Theorem 6
The Gluskabi raccordation of two signals in the kernel of the nonlinear operator $\mathbf{DN}$ with respect to a Sobolev norm (14) and connecting signals of trait θ_{initial} to θ_{final} is given by the solution of

$$\mathbf{D}^2\,\rho(\mathbf{D}^2)\,\theta_g = 0 \tag{15}$$
$$\mathbf{N}[w] = \theta_g, \tag{16}$$

subject to the gluing boundary conditions.

Proof: The type $\mathcal{T}$ is characterized by $\mathbf{DN}[w] = \mathbf{D}\theta = 0$. Thus the raccordation problem amounts to minimizing $\sum_{i=0}^{k} \rho_i (\mathbf{D}^i u)^2$ for the dynamics $\dot{\theta}_g = u$ in $\mathcal{I} = [a, b]$. The Hamiltonian for this problem is

$$H = \sum_{i=0}^{k} \rho_i \langle \mathbf{D}^i u, \mathbf{D}^i u\rangle + \langle \lambda, \mathbf{D}\theta_g - u\rangle,$$

where λ is the Lagrange multiplier we are free to choose. As before, the first order perturbation is

$$\delta H = \langle \rho(\mathbf{D}^2)u, \delta u\rangle - \langle \mathbf{D}\lambda, \delta\theta_g\rangle - \langle \lambda, \delta u\rangle.$$

Choosing λ to satisfy $\mathbf{D}\lambda = 0$, i.e., λ is constant, the stationarity condition is

$$\rho(\mathbf{D}^2)u = \lambda.$$

Combining with $\mathbf{D}\lambda = 0$ and $\mathbf{D}\theta_g = u$, this gives

$$\mathbf{D}^2\rho(\mathbf{D}^2)\theta_f,$$

since $\mathbf{D}$ commutes with the LTI $\rho(\mathbf{D}^2)$. Hence the LTI-ODE can be solved for θ_g, and finally determine w from the nonlinear ODE $\mathbf{N}[w] = \theta_g$ in $\mathcal{I}$, all with the appropriate boundary conditions. □.

Remark: The order of the linear operator (15) is $2(k+1)$, and the order of $\mathbf{N}$ is n. Hence the solution set of the above equations has $2(k+1)+n$ degrees of freedom. But $2n$ boundary conditions are needed for a smooth raccordation of w to the initial and final behaviors. This leaves $2(k+1)-n$ degrees of freedom for matching the boundary conditions

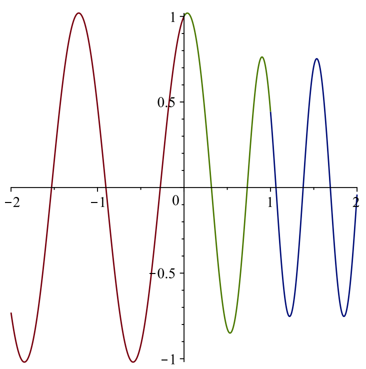

Fig. 5. Gludskabi Raccordation of harmonic type

of θ. If the least we want to do is match $\theta_g(a) = \theta_{\text{iinitial}}$ and $\theta_g(b) = \theta_{\text{final}}$, then degree of the Sobolev norm must satisfy $k \geq \frac{n}{2}$.

Example 2: Consider the general exponential type $\mathcal{T}_a = \{w | (\mathbf{D} + a)w = 0,\, a \in \mathbb{R}\}$. With parameter map $\theta = \mathbf{N}[w] = -\frac{\mathbf{D}w}{w}$. The Gluskabi raccordation $g(3\,\mathrm{e}^{-2t}, 10\,\mathrm{e}^{-t})$ in [0,1] for $k = 2$ (i.e., we take an W_2-norm) is set up as $\mathbf{D}\theta = u$, $\dot{w} + uw = 0$, and minimizing $\|u\|_2^2 + 2\|\dot{u}\|^2 + \|\ddot{u}\|^2$. This leads to the equation

$$\mathbf{D}(\mathbf{D}^2 - 1)^2 u = 0$$

with solution of the form

$$u(t) = u_0 + (u_1 + u_2 t)\cosh(t) + (u3 + u_4 t)\sinh(t).$$

Its integral, $\theta)t)$, brings in a fifth integration constant. Finally integrating $\dot{w} + \theta(t)w = 0$ gives a sixth integration constant. These can now be specified to set $u(0) = u(1) = 0$, match the exponents, $\theta(0) = -2$ and $\theta(1) = -1$ and match glue the pieces $w(0) = 3$, and $w(1) = \frac{10}{\mathrm{e}}$.

Example 3: Here we consider the type of all harmonic functions $\mathcal{T}_{\text{harm}} = \{\, w \,|\, (\mathbf{D}^2 + \omega^2)w = 0,\ \omega \in \mathbb{R}\,\}$, so that $\mathbf{N}_{\text{harm}}[w] = \frac{\mathbf{D}^2 w}{w}$. The Gluskabi raccordation in $\mathcal{I} = [0,1]$ for the L_2 norm in $\mathcal{V}$ is the solution of

$$\ddot{w} + ((1-t)\omega_-^2 + t\omega_+^2)w = 0$$

In figure 5 we show a frequency doubling raccordation for harmonic signals. Note that the square of the frequency, but not the frequency gets linearly ramped.
Here w_{final} was chosen so that the gluing is smooth. For arbitrary harmonic signals there are not enough degrees of freedom to accomplish this.

4.3 Riemannian Metric

An alternative image approach is possible. A good parameterization of the type $\mathcal{T}$ means that the map $\rho : \Theta \to \mathcal{T}$ is a bijection. Endow $\mathcal{B}|_{\mathcal{I}}$ with a metric $d_{\mathcal{T}}$ (one that is meaningful in practice). For instance, $d_{\mathcal{T}}(y, y+\delta y) = \int_{\mathcal{I}} (\delta y)^2\,\mathrm{d}t$. The inverse map ($\rho^{-1}$ is a parameter identification map) pulls this back to a metric on Θ.

$$d_{\mathcal{T}}(y, y+\delta y) = \sqrt{\mathrm{d}\theta^\top G(\theta)\,\mathrm{d}\theta} = d_\Theta(\theta, \theta + \mathrm{d}\theta),$$

with metric tensor $G(\theta)$. Now find the minimum length (under the above metric in Θ between the points θ_{initial} and θ_{final}. Minimizers of $\int \left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^\top G(\theta) \left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right) \mathrm{d}\tau$ are also geodesics by a theorem of Milnor [1963]. Note that this does not make the parameterization unique, but it makes the parameterization consistent with the given metric space $(\mathcal{T}, d_{\mathcal{T}})$. The Gluskabi raccordation is thus a homotopy in the space of behaviors. Once the geodesic path in Θ is found, one can use the parameterization ρ to (locally in time) determine the element $\rho(\theta(\tau)) = w(\cdot;\theta(\tau)) \in \mathcal{T}$. The obvious reconstruction at $t = \tau \in \mathcal{I}$ is then

$$\theta(\tau) \to w_g(\tau) = w(\tau; \rho(\tau)).$$

4.4 Harmonic type with variable frequency

Consider the type, $H_{\mathbb{C},\mathbb{R}}$, of harmonic functions of the form $x(t) = A\,\mathrm{e}^{j\omega t}$, parameterized by complex amplitude (phasor) $A \in \mathbb{C}$, and radial frequency $\omega \in \mathbb{R}$. In addition, we shall only consider the functions over the interval $[0, T]$.

It is readily established that the functional $D : H_{\mathbb{C},\mathbb{R}} \times H_{\mathbb{C},\mathbb{R}} \to \mathbb{R}_+$ given by

$$D_T(x, y) = \sqrt{\frac{1}{T}\int_0^T \|x(t) - y(t)\|^2\,\mathrm{d}t}$$

defines a distance on $H_{\mathbb{C},\mathbb{R}}$, restricted to $[0, T]$.

Denote $x(t)$ by its parameters $\phi(A, \omega)$. Then it is readily seen that for all $c \in \mathbb{C}$ and $\omega \in \mathbb{R}$, we have for all $T > 0$

$$\begin{aligned} &D_T(\phi(cA_0, \omega_0 + \omega), \phi(cA_1, \omega_1 + \omega)) \\ &= |c| D_T(\phi(A_0, \omega_0), \phi(A_1, \omega_1)) \end{aligned}$$

In particular, we see invariance w.r.t. frequency shift and homogeneity w.r.t. positive reals. This metric structure on $H_{\mathbb{C},\mathbb{R}}$ induces a Riemannian metric on the parameter space $\Theta = \mathbb{C} \times \mathbb{R}$ constructed as follows:

$$D_T(\phi(A + \mathrm{d}A, \omega + \mathrm{d}\omega), \phi(A, \omega)) = [\mathrm{d}A^*, \mathrm{d}\omega] G_{(A,\omega)}(T) \begin{bmatrix} \mathrm{d}A \\ \mathrm{d}\omega \end{bmatrix}$$

where $G_{(A,\omega)}(T)$ is the metric tensor at the point (A, ω) in parameter space.

$$G_{(A,\omega)}(T) = \begin{bmatrix} 1 & \dfrac{jAT}{2} \\ -\dfrac{jA^*T}{2} & \dfrac{|A|^2 T^2}{3} \end{bmatrix}. \qquad (17)$$

This set of metrics (one for each T) in turn satisfies, for all $k > 0$

$$G_{(A,\omega)}(T) = G_{(A/k,\omega)}(kT).$$

Hence, without loss of generality, it suffices to consider the single metric corresponding to $T = 1$, suitably rescaling frequencies of course. We simply denote then $G_{(A,\omega)}(1)$ by $G_{(A,\omega)}$.

For simplicity, we shall now consider the subspace $H_{\mathbb{R},\mathbb{R}}$ of $H_{\mathbb{C},\mathbb{R}}$, consisting of the base harmonics $\mathrm{e}^{j\omega t}$ with *real* amplitudes. Thus in $\Theta_1 = \phi(H_{\mathbb{R},\mathbb{R}})$

$$G_{(A,\omega)} = \begin{bmatrix} 1 & \\ & \dfrac{A^2}{3} \end{bmatrix}$$

We directly note the similarity of this metric with the metric tensor in $\mathbb{R}^2$, parameterized by polar coordinates (ρ, θ). Indeed,

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\theta^2.$$

In Θ, we have for the line element the length $\mathrm{d}s^2 = \mathrm{d}A^2 + \frac{A^2}{3}\mathrm{d}\omega^2$. It is crying out loudly to consider $(A, \frac{\omega}{\sqrt{3}})$ as polar coordinates in a hypothetical flat space, with $\omega/\sqrt{3}$ acting as the angular coordinate. In what follows this will be referred to as the *polar angle*. However, this flat space is not $\mathbb{R}^2$ or $\mathbb{C}$, since ω and $\omega + 2\sqrt{3}\pi$ cannot be identified. The only way to visualize this flat space is by multiple copies of $\mathbb{R}^2$ overlapping each other. This necessitates introducing a branchcut. Unlike the case of

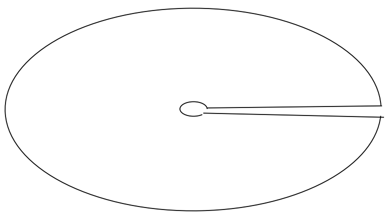

Fig. 6. A branchcut in $\mathbb{C}$

polynomial roots in complex analysis, there are an infinite number of overlapping copies of $\mathbb{C}$ in this case, 3 copies are shown in figure 7. Once it is known that the space

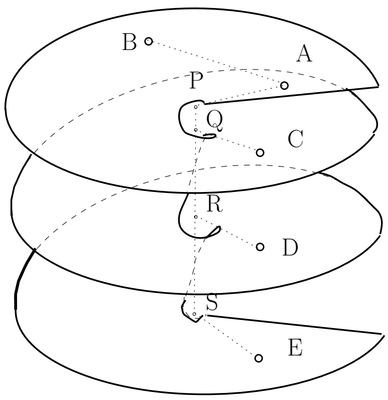

Fig. 7. These paths are geodesics

is flat, aka Euclidean, it is inferred that shortest paths must be straight lines. For instance the shortest path from point A, say with parameters (1,0) to B with parameters $(2, \pi)$, and thus corresponding polar angle $\pi/\sqrt{3}$, is the straight line AB in Θ_1. However, the $\mathbb{R}^2$ parameterization is no longer faithful when $\Delta\omega \geq \sqrt{3}\pi$, and a path must be followed that stays on Θ_1 at all times. For instance to connect A to C, corresponding to $(1, 2\sqrt{3}\pi)$, follow A to an infinitesimal circle (radius ϵ) around the origin, P, turn around to Q, and follow the radius to C. The path length is then in the limit $\epsilon \to 0$ equal to $\overline{AP} + \overline{PQ} + \overline{QC} = 1 + 0 + 1 = 2$. Likewise, we might expect that the distances from A to D and E, respectively with coordinates $(1, 4\sqrt{3}\pi)$ and $(1, 6\sqrt{3}\pi)$ are also equal to the *same* 2. However when these distances are computed by D_1 in $L_2([0, 1], \mathbb{C})$, and not in $H_{\mathbb{R},\mathbb{R}}$, one finds rather expectedly $D(A, B) = \sqrt{5}$, and not surprisingly

$$\begin{aligned} D(A, C) &= 1.477366 \\ D(A, D) &= 1.406929 \\ D(A, E) &= 1.39363. \end{aligned}$$

The paths APQC etc can therefore not be *shortest* paths in L_2, but they are in $H_{\mathbb{R},\mathbb{R}}$. In fact the distance between (1,0) and $(1, \omega)$ in L_2, i.e., D, is shown as function of ω in figure 8. It is seen to fluctuate about $\sqrt{2}$. The discrepancy

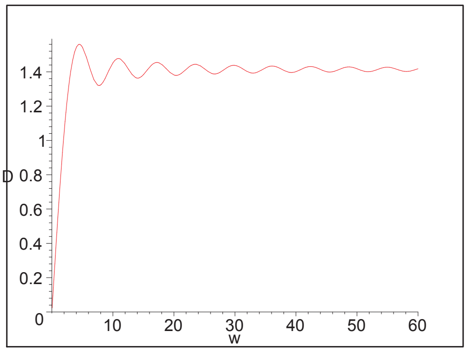

Fig. 8. Distance (1,0) to $(1, \omega)$

stems from the fact that D_T really computes the distance in the space L_2. The two-dimensional manifold $H_{\mathbb{R},\mathbb{R}}$ sits warped inside this ambient space, and the metric D_T is "as the crow flies", but then in L_2. This crow does not have to walk on $H_{\mathbb{R},\mathbb{R}}$.

This is not puzzling:. Paths connecting two harmonics with $\Delta\omega > \sqrt{3}\pi$, described above consist of parts with $\dot{\omega} = 0$ and parts where $A = 0$, at least in the limit. Are the potentially optimal? For this we check the necessary conditions for optimality. Indeed, straightforward analysis for optimization of

$$J = \int_0^T \sqrt{\dot{A}^2 + \frac{A^2}{3}\dot{\omega}^2}\,\mathrm{d}t$$

gives, upon adjoining $\dot{A} = u$ and $\dot{\omega} = \sqrt{3}v$, the Hamiltonian

$$H = \sqrt{u^2 + A^2v^2} + \lambda_A u + \lambda_\omega \sqrt{3}v.$$

This yields the optimality conditions

$$\frac{u}{\sqrt{u^2 + A^2v^2}} + \lambda_A = 0 \quad \frac{A^2v}{\sqrt{u^2 + A^2v^2}} + \sqrt{3}\lambda_\omega = 0. \tag{18}$$

and the Euler-Lagrange equations

$$\dot{\lambda_A} = -\frac{Av^2}{\sqrt{u^2 + A^2v^2}} \tag{19}$$

$$\dot{\lambda_\omega} = 0. \tag{20}$$

The two types of paths described above satisfy these necessary conditions! They are the minimal paths on $H_{(\mathbb{R},\mathbb{R})}$. The description of Θ_1 as an infinite screw does not fail. The figure 9 illustrates this 'pinching' effect on a raccordation of functions in this type.

5. GLUSKABI FILTERING/IDENTIFICATION

Given a signal $y \in \mathcal{B}$, but not of type $\mathcal{Y}$. The objective is to find signal $\hat{y} \in \mathcal{Y}$ that is "closest" to y in some sense. We focus on signal identification in $\mathcal{B} = C^k(\mathbb{R}, \mathbb{R})$, for k sufficiently large. so that all necessary derivatives are well defined. Obviously this filter theory will not encompass the usual "white noise" models, but should be seen as a deterministic form of filtering.

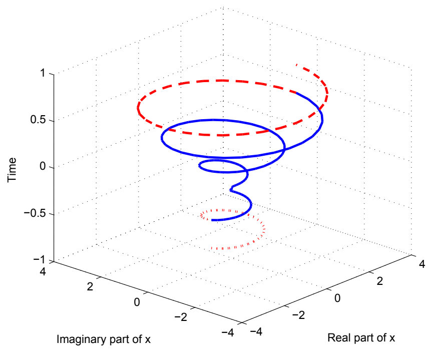

Fig. 9. Geodesic raccordation from A to C (Fig 7).

5.1 Kernel form

Suppose the type of interest is the kernel of an LTI operator

$$\mathcal{Y} = \{\, w \,|\, \widehat{\mathbf{Op}}\, w = 0\,\}, \tag{21}$$

and is observed in some unmodeled disturbance. Let the observed signal be y. Our method is akin to a *total least squares* (TLS) approach: Find $\hat{y}$ such that the modeling error $\|\widehat{\mathbf{Op}}\,\hat{y}\|$ and the discrepancy $\|y - \hat{y}\|$ are both small. Without any prejudice we take equal weights for both norms and determine $\hat{y}$ that minimizes the total misfit (to model and data)

$$J = \int_{t-1}^{t} \left(\|\widehat{\mathbf{Op}}\,\hat{y}(s)\|^2 + \|y(s) - \hat{y}(s)\|^2 \right) \mathrm{d}s. \tag{22}$$

Standard variational arguments, with $\widetilde{y}$ denoting the variation of $\hat{y}$ lead to

$$\int_{t-1}^{t} \left\{ \widehat{\mathbf{Op}}^{*}\, \widehat{\mathbf{Op}}\, \hat{y} - (y - \hat{y}) \right\}^{\top} \widetilde{y}\,\mathrm{d}s.$$

Note that if $\widehat{\mathbf{Op}} = R(\mathbf{D})$, where R is polynomial with constant coefficients, then $\widehat{\mathbf{Op}}^{*} = R(-\mathbf{D})$ is its adjoint. Hence the DuBois-Reymond lemma yields the necessary condition

$$[\widehat{\mathbf{Op}}^{*}\widehat{\mathbf{Op}} + 1]\,\hat{y} = y. \tag{23}$$

with appropriate boundary conditions at $s = t-1$ and $s = t$.

Example 1: Estimation of a signal, modeled as being constant over an interval of length 1. We have the type in kernel form

$$\mathcal{Y} = \{\, y \,|\, \mathbf{D}y = 0\,\}.$$

for which the filter is given by (note that $\mathbf{D}^* = -\mathbf{D}$)

$$(-\mathbf{D}^2 + 1)\hat{y} = y, \text{ in } [t-1, t]$$

The Green's function $G_t(s, s_0)$ is the solution to

$$(\mathbf{D}^2 - 1)G_t(s, s0) = -\delta(s - s_0),\ t-1 < s_0 < t,$$

with boundary conditions $G_t(t-1, s_0) = G_t(t, s_0) = 0$ for $t-1 < s_0 < t$.

The solution is readily found to be

$$G_t(s, s_0) = \begin{cases} \dfrac{\sinh(t - s_0)\sinh(s - t + 1)}{\sinh 1} & \text{if } t{-}1 < s < s_0 \\[2mm] \dfrac{\sinh(t - s)\sinh(s_0 - t + 1)}{\sinh 1} & \text{if } s_0 < s < t. \end{cases}$$

Thus, in explicit form, the filter has particular (data dependent) solution

$$\begin{aligned} \hat{y}_p(s) &= \int_{t-1}^{t} G_t(s, s_0) y(s_0) \mathrm{d}s_0 \\ &= \frac{\sinh(t-s)}{\sinh(1)} \int_{t-1}^{s} \sinh(s_0 - t + 1) y(s_0)\,\mathrm{d}s_0 + \\ &\quad + \frac{\sinh(s - t + 1)}{\sinh(1)} \int_{s}^{t} \sinh(t - s_0) y(s_0)\,\mathrm{d}s_0. \end{aligned}$$

Note that this is non-causal, and thus corresponds to *smoothing*. The homogeneous solution is

$$\hat{y}_h(s) = \frac{1}{\sinh(1)}[\hat{y}(t-1)\sinh(t-s) + \hat{y}(t)\sinh(s-t+1)].$$

The optimal smoother is the combination $\hat{y} = \hat{y}_p + \hat{y}_h$.

A causal form can be obtained by replacing future data, $y(s_0)$, for $s_0 > s$, in the above expression by the present estimate, $\hat{y}(s)$. This yields a causal estimator

$$\begin{aligned} \hat{y}_{\text{causal}}(s) &= \frac{\sinh(t-s)}{\sinh(1) + \sinh(t-s-1)\cosh(t-s)} \\ &\times \left[\hat{y}(t-1) + \int_{t-1}^{s} \sinh(s_0 - t + 1) y(s_0)\,\mathrm{d}s_0 \right], \end{aligned}$$

preserving the trait of the signal class.

Example 2: If the signal is known to be harmonic, with frequency ω, then $R(\xi) = \xi^2 + \omega^2$, yields the filter

$$[(\mathbf{D}^2 + \omega^2)^2 + 1]\hat{y} = y$$

which again can be solved via the Green's function technique.

More generally, we have :

Theorem 7
Given a type specified as the kernel of a nonlinear operator,

$$\mathcal{Y} = \{\, y \in \mathcal{B} \,|\, \mathbf{N}[y] = 0\,\},$$

and a signal $y \notin \mathcal{Y}$, then the signal $\hat{y} \in \mathcal{Y}$ closest to y in the sense of minimizing

$$J = \int_{t-1}^{t} \left\{ \|N(\mathbf{D}, \hat{y})\|^2 + \|\hat{y} - y\|^2 \right\} \mathrm{d}s \tag{24}$$

is the solution to

$$\nabla \mathbf{N}^*[\mathbf{N}[\hat{y}]] + \hat{y} = y. \tag{25}$$

Proof: Compute the first variation of the performance index as follows: Let the nonlinear operator have highest derivative $\mathbf{D}^n$, then $\mathbf{N}(\mathbf{D})[w] = \mathcal{N}(w, \mathbf{D}w, \ldots, \mathbf{D}^n w)$. Consider the multivariable function $\mathcal{N}(\xi_0, \xi_1, \ldots, \xi_n) = \mathcal{N}(\overline{\xi})$ which represents the operator. Note that

$$\begin{aligned} \|\mathcal{N}(\overline{\xi} + \widetilde{\xi})\|^2 - \|\mathcal{N}(\overline{\xi})\|^2 &= \langle \mathcal{N}(\overline{\xi}), \nabla\mathcal{N}(\overline{\xi})\widetilde{\xi} \rangle \\ &= \langle [(\nabla\mathcal{N})^*\mathcal{N}](\overline{\xi}), \widetilde{\xi} \rangle. \end{aligned}$$

Returning to the operator form in $\mathbf{D}$, we get thus

$$\delta \int_{t-1}^{t} \|\mathbf{N}[\hat{y}]\|^2\,\mathrm{d}s = \int_{t-1}^{t} \nabla\mathbf{N}^*[\mathbf{N}[\hat{y}]]\,\delta\hat{y}\,\mathrm{d}s$$

Combined with the linear term $\int_{t-1}^{t}(\hat{y} - y)\,\delta\hat{y}\,\mathrm{d}s$ using again by the Dubois-Reymond lemma this proves (25). $\square$

Example 4: It is desired to filter a signal y which is only known to be of first order (i.e., satisfies $\widehat{\mathbf{Op}}\, y = R(\mathbf{D}) = 0$ for some first order polynomial $R(\xi)$). Here

$$\mathcal{Y} = \{\, w \mid \ddot{w}w - \dot{w}^2 = 0\,\},$$

Thus, $\mathcal{N}(\xi_0, \xi_1, \xi_2) = \xi_0\xi_2 - \xi_1^2$. We find

$$\nabla\mathcal{N} = [\xi_2, -2\xi_1, \xi_0]$$

so that $\nabla\mathbf{N}[w] = [\mathbf{D}^2 w, -2\mathbf{D}w, w]$. Finally

$$[\nabla\mathbf{N}^*[\mathbf{N}[w]] = (\ddot{w} + 2\mathbf{D}\dot{w} + \mathbf{D}^2 w)(\ddot{w}w - \dot{w}^2),$$

where $\mathbf{D}$ acts on the string to its right. The optimal filter in $[t-1, t]$ is the solution to

$$(\ddot{\hat{y}} + 2\mathbf{D}\dot{\hat{y}} + \mathbf{D}^2\hat{y})(\ddot{\hat{y}}\hat{y} - \dot{\hat{y}}^2) + \hat{y} = y,$$

in $[t-1, t]$, where the $\mathbf{D}$ acts on anything following it. This is 4th order nonlinear ODE, driven by the observed signal.

5.2 Image Representation

Associated with a linear kernel representation $\widehat{\mathbf{Op}}w = 0$, the image representation in terms of the signal modes is given by the linear combination $w(t) = \sum_i \alpha_i\phi_i(t) = \overline{\alpha}^\top\Phi$. Since for the exact representation of $y(t)$, the parameter vector $\overline{y}$ is constant, we shall require for the inexact form a small variation. Thus motivated, we minimize

$$\|y - \Phi^\top\overline{y}\|^2 + \|\mathbf{D}\overline{y}\|^2 \qquad (26)$$

where the norm is the 2-norm over the interval $[t-1, t]$. The variational equation leads to the filter solution (for a scalar signal)

$$\hat{y}(t) = \Phi(t)^\top\overline{y} = (\Phi^\top(t)\Phi(t))^{-1}\Phi^\top(t)(\mathbf{D}^2 - 1)y(t). \qquad (27)$$

6. DYNAMIC GLUSKABI RACCORDATIONS

This section is devoted to the study of the dynamical raccordation case, i.e., when the (multi-dimensional) trajectories in the base behavior are constrained by the *dynamics* of a system. Since one is not allowed to step out of this base behavior, the dynamical system constraints can be called "*hard constraints*", as opposed to the type constraints which can be violatedabd are thus "*soft constraints.*" In the following sections, the Gluskabi extension is derived for linear time invariant dynamical systems.

6.1 Continuous-Time LTI Systems

Consider an LTI system and types that are described by the state space form

$$(\mathbf{D} - A)x = Bu, \quad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}.$$

A more general exposition for polynomial matrix fraction descriptions is presented in Memon [2014]. Here we take for the base behavior the set of smooth trajectories of the LTI dynamical system,

$$\mathcal{B} = \{w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \mid w = \begin{bmatrix} x \\ u \end{bmatrix},\ (\mathbf{D} - A)x - Bu = 0\,\} \qquad (28)$$

where x and u are called the state and input respectively. We also use the same notation for an operator acting on a scalar signal and its extension to a vector (acting on its component). We also use the notation $\mathbf{Q}$ to denote $\rho(-\mathbf{D}^2)$, where ρ is a polynomial with positive coefficients, to indicate the Sobolev norm associated operator. Note that $\mathbf{Q}$ is selfadjoint. In what follows, we generalize slightly, but one can let $\mathbf{Op}^{\text{hard}} = [\mathbf{D} - A, -B]$.

Theorem 8
Given a controllable behavior specified by $\mathbf{Op}^{\text{hard}}$ and a type, $\mathcal{T}$, described by an LTI $\widehat{\mathbf{Op}}$. The trajectories in the Gluskabi extension with respect to the Sobolev norm $\|\cdot\|_{\mathbf{Q}}$, where $\mathbf{Q} = \rho(\mathbf{D}^2)$ are the solutions to

$$\mathbf{Op}^{\text{hard}}\mathbf{Op}^{\text{hard}^*}\lambda = 0 \qquad (29)$$

$$\widehat{\mathbf{Op}}^*\mathbf{Q}\widehat{\mathbf{Op}}\, w = -\mathbf{Op}^{\text{hard}^*}\lambda \qquad (30)$$

Proof: The dynamic Gluskabi problem seeks to minimize $\|\widehat{\mathbf{Op}}\, w\|_{\mathbf{Q}}$ such that $\mathbf{Op}^{\text{hard}}w = 0$ over all $w \in \mathcal{B}$. The Hamiltonian for this problem is

$$H = \frac{1}{2}\langle\mathbf{Q}\,\widehat{\mathbf{Op}}w, \widehat{\mathbf{Op}}w\rangle + \langle\lambda, \mathbf{Op}^{\text{hard}}w\rangle.$$

The first variation gives

$$\delta H = \langle\widehat{\mathbf{Op}}^*\mathbf{Q}\widehat{\mathbf{Op}}\, w, \delta w\rangle + \langle\mathbf{Op}^{\text{hard}^*}\lambda, \delta w\rangle.$$

Choosing the Lagrange multipliers as solution to

$$\widehat{\mathbf{Op}}^*\mathbf{Q}\widehat{\mathbf{Op}}\, w = -\mathbf{Op}^{\text{hard}^*}\lambda$$

avoids computing δw. Since also $\mathbf{Op}^{\text{hard}}\, w = 0$, we find that λ solves

$$\mathbf{Op}^{\text{hard}}\mathbf{Op}^{\text{hard}^*}\lambda = 0$$

as the LTI operators commute. □
A compact form of the Hamiltonian system is

$$\begin{bmatrix} \widehat{\mathbf{Op}}^*\mathbf{Q}\widehat{\mathbf{Op}} & -\mathbf{Op}^{\text{hard}^*} \\ \mathbf{Op}^{\text{hard}} & 0 \end{bmatrix}\begin{bmatrix} w \\ \lambda \end{bmatrix} = 0. \qquad (31)$$

7. RACCORDATION OF PERIODIC OPERATIONS.

7.1 Case of equal periods

We consider the system represented in state space form $\dot{x} = Ax + Bu$. Suppose we want to connect two periodic regimes $x_0(t)$ and $x_1(t)$ of a common frequency ω_0 on the interval $[t_0, t_1]$. As mentioned earlier, the initial and final periodic $x_i(t)$ where $i = 0, 1$ can be approximated by solutions of

$$\mathbf{D}\prod_{k=1}^{N}\left((\mathbf{D}^2 + k^2\omega_0^2)I\right)x_i = 0 \qquad (32)$$

It is exact if the signals to be connected only have N harmonic components. We make this assumption as it simplifies analysis and further the Fourier coefficients of $x_i(t)$ go to zero as $n \to \infty$ by the Riemann-Lebesgue lemma (See Folland [1999]). We also assume that the signals have the same DC Fourier component, equal to the average value of both signals across one period is the same.
Hence the operator that defines the trait is $\widehat{\mathbf{Op}}(x, u) = \left(\prod_{k=1}^{n}(\mathbf{D}^2 + k^2\omega_0^2)I\right)x$. Let us define the differential operator

$$a(\mathbf{D}) = \prod_{k=1}^{N}(\mathbf{D}^2 + k^2\omega_0^2)I \qquad (33)$$

We emphasize that the $x_i(t)$ $(i = 1, 2)$ are solutions of the dynamical system,

$$[\mathbf{D} - A(t)\ \ B(t)] \begin{bmatrix} x_i(t) \\ u_i(t) \end{bmatrix} = 0$$

To find a raccordation, we are thus led to the following optimization problem:

$$\min_{w(t)} J(w) = \min_{w(t)} \int_{t_0}^{t_1} (a(\mathbf{D})x)^\top (a(\mathbf{D})x) dt \tag{34}$$

where $w = [x^\top, u^\top]^\top$ and subject to the constraints

$$[\mathbf{D} - A(t), -B(t)]\, w = 0. \tag{35}$$

and the boundary conditions

$$\sigma_{t_0}(\mathbf{D}^{(j)} x) = \sigma_{t_0}(D^{(j)} x_0). \tag{36}$$

$$\sigma_{t_1}(\mathbf{D}^{(j)} x) = \sigma_{t_1}(D^{(j)} x_1). \tag{37}$$

for $j = 1, 2, ...2N - 1$.
These gluing conditions are necessary to ensure the required smoothness of the raccordation. In the definition of the behavior given in the previous section, we required our functions $w(t)$ in the behavior to be C^{2N-2}. By Theorem 8, the first order necessary conditions for optimality are obtained by solving

$$\begin{bmatrix} (\mathbf{D} - A) & -B & 0 \\ a(\mathbf{D})^2 & 0 & -(\mathbf{D} + A)^\top \\ 0 & 0 & -B^\top \end{bmatrix} \begin{bmatrix} x \\ u \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{38}$$

Note that here $a(\mathbf{D})$ is self-adjoint.

7.2 Case of different periods

Suppose that the two periodic regimes, $x_0(t)$ and $x_1(t)$, to be connected have different base frequencies, ω_0 and ω_1 respectively. Let us define now,

$$a(\mathbf{D}, \omega(t)) = \left(\prod_{k=1}^{N} (\mathbf{D}^2 + k^2 \omega^2(t)) \right). \tag{39}$$

Let there not be a soft constraint on the input u. The solution of the raccordation problem, $x(t)$, should be such that $a(\mathbf{D}, \omega(t))x(t)$ is close to zero, but we also want to ensure that $\omega(t)$ changes slowly. Since $\omega(t)$ is a function of time, we have to augment the behavior by defining $\hat{w} = [w^\top \omega]^\top$. The operator $\overline{\mathbf{Op}}$ that defines the trait is now given by

$$\overline{\mathbf{Op}}(\hat{w}) = \begin{bmatrix} a(\mathbf{D}, \omega)x \\ \mathbf{D}\omega \end{bmatrix} \tag{40}$$

where also that $w = [x^\top, u^\top]^\top$. The trajectories to be connected in the extended behavior are $\hat{w}_0(t) = [x_0^\top(t), \omega_0]^\top$ and $\hat{w}_1(t) = [x_1^\top(t), \omega_1]^T$. We arrive at the optimal control problem,

$$\min_{w,\omega} J(w, \omega) = \frac{1}{2} \left(\langle a(\mathbf{D}, \omega)x, a(\mathbf{D}, \omega)x \rangle + \langle \mathbf{D}\omega, \mathbf{D}\omega \rangle \right) \tag{41}$$

subject to the gluing constraints for x and ω. Proceeding as before, a term $\langle D\omega, D\omega \rangle$ must be added to the Hamiltonian in the previous problem. Computing the Gateaux variation yields

$$\begin{bmatrix} (\mathbf{D} - A(t)) & -B(t) & 0 \\ a(\mathbf{D}, \omega)^* a(\mathbf{D}, \omega) & 0 & -(D + A(t))^\top \\ 0 & 0 & -B(t)^\top \end{bmatrix} \begin{bmatrix} x \\ u \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{42}$$

where ω satisfies an ODE

$$\mathbf{D}^2 \omega = f(x, Dx, ..D^{2n-1}x, \omega, D\omega, ...D^{2n-1}\omega) \tag{43}$$

Here $a(D, \omega)^*$, the adjoint operator, *acts* on $a(\mathbf{D}, \omega)$ and a^*a is therefore not simply the commutative product of the time-variant differential operators. For the case where both x and u are soft-constrained, see Murali and Verriest [2017]. Finally, it should be noted that this problem is relevant for the frequency swept identification method of LTI systems (See Pintelon and Schoukens [2001]).

An alternative approach, based on the characterization of periodicity by the shift operator is presented in Verriest [2021]. This method is exact, however, because of the infinite dimensionality the boundary conditions for the Gluskabi raccordation are no longer local and require the matching of function segments.

8. SUMMARY AND BEYOND

A theoretical framework was introduced for a definition of gracefulness in transitions, for signals and for system behaviors. It was related to the notion of persistence of behavior, for which a precise definition was given. Further work, not reported here, involves transitions between periodic regimes of driven systems in Yeung and Verriest [2010, 2011], limit cycles for smooth nonlinear systems in Verriest and Murali [2018] and for hybrid systems in Murali et al. [2018, 2019, 2020]. Behavior transitions for a jumper on a trampoline are described in Murali and Verriest [2020] and for a rimless wheel in Murali and Verriest [2021].

ACKNOWLEDGEMENTS

The author thanks the Giovanni-Prodi program of the Institute für Mathematik, Universität-Würzburg, Würzburg, Germany, for its support and hospitality while part of this work was performed.

REFERENCES

Andresen, B., Berry, R.S., Nitzan, A., and Salamon, P. (1977). Thermodynamics in finite time. i. the step-carnot cycle. *Physical Review A*, 15(5), 2086.

Berry, R.S., Kazakov, V., Sieniutycz, S., Szwast, Z., and Tsirlin, A. (2000). *Thermodynamic optimization of finite-time processes.* Wiley Chichester.

Bostan, A. and Dumas, P. (2010) Wronskians and linear independence. The American Mathematical Monthly, Vol. 117, No. 8. pp. 722-727.

De Vos, A. and Desoete, B. (2000). Equipartition Principles in Finite-time thermodynamics. *Journal of Non-Equilibrium Thermodynamics*, 25(1), pp. 1–13.

Fan T.D.and Hao, D.N. (1994). *Differential Operators of Infinite Order with Real Arguments and Their Applications.* World Scientific.

Folland, G.F. (1999). *Real Analysis: Modern Techniques and Their Applications.* Wiley.

Golubitsky, M. and Stewart, I. (2003). *The symmetry perspective: from equilibrium to chaos in phase space and physical space*, volume 200. Springer.

Haynes, G.C., Rizzi, A.A. and Koditschek, D.E. (2012). Multistable phase regulation for robust steady and transitional legged gaits. *International Journal of Robotics Research.* 31(14), pp. 1712–1738.

Memon, A.-B. (2014) Graceful connections in dynamical systems–an approach to gait transitions in robotics Ph.D. Dissertation, Georgia Institute of Technology.

Memon, A.B. and Verriest, E.I. (2014a). Kernel Representation Approach to Persistence of Behavior. *IFAC Proceedings* Vol.47, Issue 3, pp. 6032–6037.

Memon, A.B. and Verriest, E.I. (2014b). Graceful Gait Transitions for Biomimetic Locomotion – The Worm. *Proceedings of the 53rd IEEE Conference on Decision and Control*, pp. 2958–2963, Los Angeles, CA.

Milnor, J. (1963) *Morse Theory* Princeton University Press

Murali, V. (2021). *Graceful transitions between periodic motions for nonlinear and hybrid systems.* Ph.D. Dissertation Georgia Tech.

Murali, V., Ames, A.D. and. Verriest, E.I. (2019). Optimal Walking Speed Transitions for Fully Actuated Bipedal Robots. *Proceedings of the 57-th IEEE Conference on Decision and Control*, pp. 6295–6300, Nice, France.

Murali, V., Hyun, N.-s.P. and Verriest, E.I. (2018). Graceful Gait Transitions for Hopping Robots on Deformable (Granular and Elastic) Terrain. *Proceedings of the 2018 American Control Conference*, pp. 1299–1304, Milwaukee, WI.

Murali, V. Hyun, N.-s.P and Verriest, E.I. (2020). Graceful Transitions between Periodic Walking Gaits of Fully Actuated Bipedal Robots. *Proceedings of the 2020 American Control Conference,* Denver CO. pp. 1043–1048.

Murali, V. and Verriest, E.I. (2017). Raccordation between Periodic Signals. *IFAC-PapersOnLine* Volume 50, Issue 1, pp. 2211–2216.

Murali, V. and Verriest, E.I. (2018). Graceful Transitions between Periodic Orbits of Parametrized Nonlinear Systems. *Proceedings of the 57-th IEEE Conference on Decision and Control*, pp. 397–402 Miami-Beach, FL.

Murali, V. and Verriest, E.I. (2021a). Periodic Behaviors and Graceful Transitions between Them for Bimodal Systems: Example of a Jumper on a Trampoline. *IEEE Cntrol System Letters.*

Murali, V. and Verriest, E.I. (2021b). Transitions for the Rimless Wheel on Flat Terrain. Proceedings of the 60th IEEE Conference on Decision and Control, Austin, TX, pp. 4660-4663.

Pintelon R. and Schoukens, J. (2001). *System Identification: A frequency domain approach.* IEEE Press, 2001.

Polderman, J.W. and Willems, J.C.. (1998). *Introduction to Mathematical System Theory* Springer Verlag.

Sato, M., Kashiwara, M. and Kawai, T. (1983). Linear Differential Equations of Infinite Order and Theta Functions *Advances in Mathematics*, Vol. 47, pp. 300-325.

Silverman, R.A. (1984). *Introductory complex analysis.* Dover Publications.

Sultan, C. (2007). Nonlinear Systems Control Using Equilibrium Paths. *Proceedings of the 46-th IEEE Conference on Decision and Control*, New Orleans, LA, pp. 50–55.

Sultan, C. (2008). Simple Conditions for Arbitrarily Close Deployment and Equilibrium Paths and Quasistationary Deployment. *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico, pp. 1175–1179.

Sultan, C. and Kalmar-Nagy, T. (2011). Graceful Passage Through Hopf Bifurcation. *IFAC-Proceedings* Volume 44, Issue 1, pp. 10899–10903.

Verriest, E.I. (2012). Persistence of Behavior. *Proceedings of the 2012 MTNS*, Melbourne Australia. `http://fwn06.housing.rug.nl/mtns2012/FullPaper/MTNS2012_0246_paper.pdf`.

Verriest, E.I. (2021) Maximum Persistence of Periodic Behavior leads to Functional Equations. IFAC-PapersOnLine Volume 54, Issue 18, Pages 90-95.

Verriest, E.I. and Murali, V. (2018). Graceful transitions between limit cycles of a parameterized system. *Proceedings of the 2018 American Control Conference*, pp. 6075–6080, Milwaukee, WI.

Verriest, E.I. and Yeung, D. (2008). Maximally Smooth Transfers: Gluskabi Raccordations. *Proc. 17th International Symposium on Mathematical Theory of Networks and Systems*, Blacksburg, VA, `https://scholar.lib.vt.edu/MTNS/Papers/139.pdf`.

Willems, J.C. (2007). The behavioral approach to open and interconnected systems. *Control Systems Magazine*, IEEE, 27(6), pp. 46–99.

Yeung, D. (2011). Maximally smooth transition: the Gluskabi raccordation. Ph.D. Dissertation, Georgia Tech.

Yeung, D. and Verriest, E.I. (2009). On Connecting Trajectories with Maximum Persistence of Behavior. *Proceedings of the 48th Conference on Decision and Control and 28th Chinese Control Conference*, pp. 5039–5044, Shangai, PRC.

Yeung, D. and Verriest, E.I. (2010). On Connecting Periodic Trajectories with Quasi- Periodic Paths. *Proceedings of the 49-th IEEE Conference on Decision and Control*, pp. 4825–4830.

Yeung, D. and Verriest, E.I. (2011). Smooth Transitions via Quasi-Periodic Paths. *IFAC Proceedings Volumes* Vol 44, Issue 1, pp. 5549-5554.